Fermi National Accelerator Laboratory

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HARD AMPLITUDES IN GAUGE THEORIES [1]

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Abstract

In this lecture series I present recent developments in perturbation theory methods for gauge theories for processes with many partons. These techniques and results are useful in the calculation of cross sections for processes with many final state partons which have applications in the study of multi-jet phenomena in high-energy colliders. The results illuminate many important and interesting properties of non-abelian gauge theories.

1 Introduction

In high energy collisions among hadrons and/or leptons the production of final states with a large number of energetic, widely separated partons gives rise to events with many jets in the final state potentially important probe on new physics [1], *e.g.* in the case of the sequential decays of new heavy particles, such as a Higgs decaying to four jets through real W/Z pairs, or such as a pair of heavy gluinos decaying into a multi-jet system through a chain-decay of the various unstable supersymmetric particles. The possibility of using these observables to identify new phenomena relies on our capability to predict the production rates and features of the standard multi-jet production mechanisms which often provide a significant background to these discovery channels.

Consider the production of the top quark at a high energy hadron collider,

$$p + \bar{p} \longrightarrow t + \bar{t} + x.$$

Now each top quark decays in the standard model via the following process,

$$t \longrightarrow W + b,$$

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followed by the decay of the W boson into leptons or quarks.

$$
\begin{aligned}
W \quad &\longrightarrow l + \bar{l} \\
&\longrightarrow q + \bar{q}.
\end{aligned}
$$

Hence there are three signals through which the top quark can be observed,

- $l + \bar{l}' + b + \bar{b}$ + missing transverse energy,
- $l +$ two jets $+ b + \bar{b}$ + missing transverse energy,
- four jets $+ b + \bar{b}$.

The first of these has the smallest signal rate and smallest background, the next has larger rate and background, whereas the final topology has an enormous background. The calculation of these backgrounds is where the techniques developed here are most useful. What is the QCD/EW background to the second top signal? *i.e.* What is the W plus four jet production cross section at hadron colliders? For the last top pair signal, suppose the experimentalists tag on one of the b-quarks. How does the signal to background now compare? For this you need to know the six jet production rate from QCD with a $b\bar{b}$ pair.

In this series of lectures I review the recent developments for the calculation of multi-parton matrix elements in non-abelian gauge theories. Most of what is discussed here plus references can be found in more detail in the review by Mangano and Parke [2].

In Section 2 I describe the helicity-amplitude technique and introduce explicit parametrizations of the polarization vectors in terms of massless spinors. An explicit QED example is given. Then I show how to decompose the color structures of non-abelian gauge theories, first for quark-gluon amplitudes and then for pure gluonic amplitudes. Many explicitly results are presented.

Section 3 the factorization properties of the sub-amplitudes are described. The results contained in this Section are useful for a better understanding of the structure of multi-parton amplitudes in gauge theories.

Section 4 introduces the Berends-Giele recursion relations, which allows the calculation of the matrix elements to be performed in a recursive fashion, providing an algebraic algorithm which can be efficiently used for numerical evaluation of higher order processes. The usefulness of this technique is shown using W plus jet production in Hadron Colliders.

Section 5 describes the various approximation techniques that have been invented followed by the conclusions and an appendix on spinor calculus and the color truncated Feynman rules for non-abelian gauge theories.

2 Helicity Amplitudes

The use of helicity amplitudes for the calculation of multi-parton scattering in the high-energy (massless) limit was pioneered in papers by J.D. Bjorken and M. Chen [3], and by O. Reading-Henry [4], and later further developed and fully exploited by the Calkul Collaboration in a classical set of papers [5, 6].

This method relies on two important techniques;

- the decomposition of the amplitude into appropriate gauge invariant sub-amplitudes,
- and the evaluation of each gauge invariant sub-amplitude using an explicit representation of the polarization vectors and spinors for the external particles.

2.1 Polarization Vectors and Spinors

Let us start with a simple application of the helicity amplitude technique in QED. Consider the process of massless electron-positron annihilation into a photon pair. Two diagrams contribute to the process – t-channel and u-channel fermion exchange (see Figure 1). If $q, \bar{q}$ are the momenta of the electron and positron, $p_{1,2}$ are the momenta of the two photons. Then the contributions of the two diagrams are as follows:

$$M_t = \frac{-ie^2}{S_{qp_1}}\bar{u}(q)\hat{\epsilon}(p_1)(\hat{q}+\hat{p}_1)\hat{\epsilon}(p2)v(\bar{q}), \tag{2.1}$$

$$M_u = \frac{-ie^2}{S_{qp_2}}\bar{u}(q)\hat{\epsilon}(p_2)(\hat{q}+\hat{p}_2)\hat{\epsilon}(p1)v(\bar{q}). \tag{2.2}$$

Where I have used the convention that all particles are labelled (momentum, helicity, etc) as if they are outgoing so that $\sum p_i \equiv 0$, $S_{pq} \equiv (q+p)^2 = 2p\cdot q$ and $\hat{p} \equiv p_\mu\gamma^\mu$.

To check that the sum of these two diagrams form a gauge invariant subamplitude (as it must be for this simple case) replace $\epsilon(p_1) \rightarrow p_1$ then $M_t + M_u$ must be equal zero. This could also have been done with the second photon. This check of gauge invariance can be used also in the non-abelian cases considered later.

To calculate the helicity amplitudes we need projections of the spinors and the polarization vectors for both positive and negative helicity. First for the spinors, we use a Weyl basis and write

$$|p\pm\rangle \equiv \frac{1}{2}(1\pm\gamma_5)v(p) \qquad \langle p\pm| \equiv \bar{u}(p)\frac{1}{2}(1\mp\gamma_5). \tag{2.3}$$

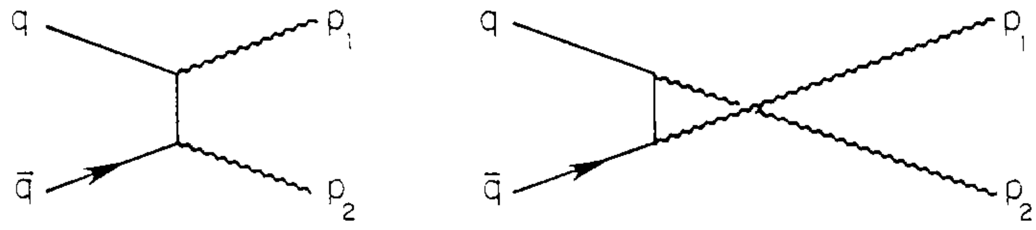

Figure 1: The abelian diagrams contributing to fermion-antifermion annihilation into two vectors.

From these spinors we can form various spinor products

$$\langle pq\rangle \;\equiv\; \langle p-|q+\rangle \qquad [pq] \;\equiv\; \langle p+|q-\rangle, \tag{2.4}$$

$$\langle p-|q-\rangle = 0 \qquad \langle p+|q+\rangle = 0 \tag{2.5}$$

which have many useful properties, which are summarized in the appendix. Here I want to emphasize one important property that both $\langle pq\rangle$ and $[pq]$ are complex square roots of S_{pq}.

We can use these spinors to form the helicity projections for the vector particles following Xu, Zhang and Chang in Ref.[7]

$$\epsilon^{\pm}_{\mu}(p,k) \;=\; \pm\frac{\langle p\pm|\gamma_\mu|k\pm\rangle}{\sqrt{2}\langle k\mp|p\pm\rangle}. \tag{2.6}$$

Where the momentum, k, is an arbitrary light-like momentum not parallel to p. These polarization vectors satisfy the standard properties;

$$\epsilon^{\pm}_{\mu}(p,k) \;=\; (\epsilon^{\mp}_{\mu}(p,k))^{*}, \tag{2.7}$$

$$\epsilon^{\pm}(p,k)\cdot p \;=\; 0, \tag{2.8}$$

$$\epsilon^{\pm}(p,k)\cdot\epsilon^{\pm}(p,k) \;=\; 0, \tag{2.9}$$

$$\epsilon^{\pm}(p,k)\cdot\epsilon^{\mp}(p,k) \;=\; -1 \tag{2.10}$$

as well as

$$\epsilon^{\pm}(p,k)\cdot k \;=\; 0. \tag{2.11}$$

The fact that this choice can be made is most easily understood from SUSY; under a supersymmetry transformation, $\delta A^\mu \rightarrow \bar{\psi}(p)\ \gamma^\mu\ v$, where v is an arbitrary spinor.

The contraction of these polarization vectors with γ_μ gives

$$\epsilon^{\pm}(p,k)\cdot\gamma \;=\; \pm\frac{\sqrt{2}}{\langle k\mp|p\pm\rangle}\left(|p\mp\rangle\langle k\mp| + |k\pm\rangle\langle p\pm|\right) \tag{2.12}$$

and a change in reference momentum leads to

$$\epsilon^{\mp}_{\mu}(p,k) \rightarrow \epsilon^{\mp}_{\mu}(p,k') - \sqrt{2}\frac{\langle kk'\rangle}{\langle kp\rangle\langle k'p\rangle}p_\mu. \tag{2.13}$$

Note that the factor in front of the right hand side polarization vector is just unity, not some complicated phase factor. This is because of the choice of normalization in the definition of $\epsilon(p,k)$. Also, since we will always evaluate gauge invariant sub-amplitudes the second term gives no contribution. Therefore we can use different reference momentum for different particles and different reference momentum for different sub-amplitudes.

Using this machinery we can now calculate $M = M_t + M_u$ for the various helicity projections for our simple electron-positron annihilation into two photon example. For a negative helicity electron and a positive helicity positron we have

$$M(q^-,\gamma_1,\gamma_2,\bar{q}^+) \;=\; \frac{(-ie^2)}{S_{qp_1}}\langle q-|\hat{\epsilon}(p_1)(\hat{q}+\hat{p}_1)\hat{\epsilon}(p_2)|\bar{q}-\rangle \tag{2.14}$$

$$+\; \frac{(-ie^2)}{S_{qp_2}}\langle q-|\hat{\epsilon}(p_2)(\hat{q}+\hat{p}_2)\hat{\epsilon}(p_1)|\bar{q}-\rangle \tag{2.15}$$

Now if we substitute our polarization vectors for the photons choosing $k_1 = k_2 = q$ for both positive or negative helicity we easily obtain

$$M(q^-,\gamma_1^+,\gamma_2^+,\bar{q}^+) \;=\; M(q^-,\gamma_1^-,\gamma_2^-,\bar{q}^+) \;=\; 0. \tag{2.16}$$

For opposite helicity photons, again choosing $k_1 = k_2 = q$, it is easy to calculate that

$$M(q^-,\gamma_1^-,\gamma_2^+,\bar{q}^+) \;=\; -2ie^2\frac{\langle q1\rangle^3\,\langle\bar{q}1\rangle}{\langle q1\rangle\langle 1\bar{q}\rangle\langle q2\rangle\langle 2\bar{q}\rangle}. \tag{2.17}$$

Of course, because of gauge invariance, the answer is independent of which reference momentum we choose for the photons but the ease of calculation is not.

Summing over all helicity amplitudes we obtain

$$|M|^2 \;=\; 8e^4\frac{u^2+t^2}{ut} \tag{2.18}$$

which is the well known results.

The real usefulness of these particular polarization vectors is only seen in non-abelian theories where the Feynman diagrams contain many $\epsilon \cdot \epsilon'$ factors. If the two vectors have the same helicity then

$$\epsilon^+(p,k) \cdot \epsilon^+(p',k') = \frac{[pp']\langle k'k\rangle}{\langle kp\rangle\langle k'p'\rangle}, \tag{2.19}$$

$$\epsilon^-(p,k) \cdot \epsilon^-(p',k') = \frac{\langle pp'\rangle[k'k]}{[kp][k'p']}. \tag{2.20}$$

If $k = k'$, both these dot products vanish. Suggesting the rule that if two vectors have the same helicity one should use the same reference momentum.

If the two vectors have opposite helicity then

$$\epsilon^+(p,k) \cdot \epsilon^-(p',k') = \frac{[pk']\langle p'k\rangle}{\langle kp\rangle[k'p']}. \tag{2.21}$$

Again, if $k' = p$ or $k = p'$ then this dot product vanishes. Suggesting the rule that for unlike helicities the reference momentum of one of the vectors should be the momentum of the other vector. Unfortunately this rule and the one above are inconsistent but they can be applied so as to minimize the number of non-zero $\epsilon \cdot \epsilon'$.

2.2 Quark-Gluon Color Decomposition

Consider a quark and an antiquark with colors α and $\bar{\alpha}$ respectively then we write the amplitude as

$$\mathcal{M}_{\bar{q}qng} = \sum_{perm} (\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n})_{\alpha\bar{\alpha}}\, m(q,\bar{u}_q;\ p_1,\epsilon_1;\cdots;\ p_n,\epsilon_n,\bar{q},v_{\bar{q}}), \tag{2.22}$$

where the sum, *perm*, is over all n! permutation of the gluons. This expansion of the quark amplitude in terms of this color basis is well known and in particular was used by Kunszt in Reference [8]. We will call the color basis in Equation (2.22) the *quark dual basis*[9],[10].

For the amplitude squared,

$$\sum_{colors} |\mathcal{M}_{\bar{q}qng}|^2 = N^{n-3}(N^2-1) \sum_{\{1,\ldots,n\}} \{|m_{\bar{q}q}(q,1,\cdots,n,\bar{q})|^2 + \mathcal{O}(N^{-2})\}. \tag{2.23}$$

Notice the exponent of the leading power of N. The explicit form of the sub-leading terms for $n = 2,3,4$ is given in reference [10].

Consider quark-antiquark scattering into two gluons as a simple example. There are three diagrams which contribute to this processes, the two QED diagrams of

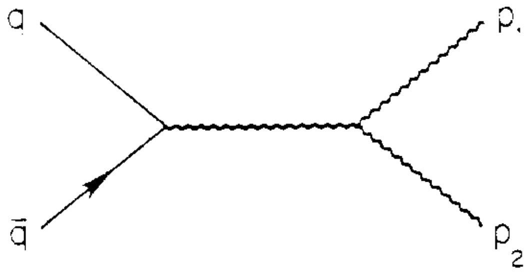

Figure 2: The non-abelian diagram contributing to fermion-antifermion annihilation into two vectors.

Fig. 1 plus the s-channel diagram, Fig. 2, which is purely non-abelian in nature. The color structures of the t,u and s-channel diagrams are

$$(\lambda^{a_1}\lambda^{a_2})_{\alpha\bar\alpha} \quad ; \qquad (\lambda^{a_2}\lambda^{a_1})_{\alpha\bar\alpha}$$

$$\lambda^{a}_{\alpha\bar\alpha}\, f^{aa_1a_2} \;=\; \frac{-i}{\sqrt{2}}[(\lambda^{a_1}\lambda^{a_2})_{\alpha\bar\alpha} - (\lambda^{a_2}\lambda^{a_1})_{\alpha\bar\alpha}] \tag{2.24}$$

respectively. Our normalization of λ^a is $tr(\lambda^a\lambda^b) = \delta^{ab}$, hence $\sqrt{2}$ in the last equation.

Therefore for the color factor $(\lambda^{a_1}\lambda^{a_2})_{\alpha\bar\alpha}$ we have a contribution from both the t- and s- channel diagrams. For a negative helicity quark and positive helicity antiquark the subamplitude is given by

$$m(q^-, g_1, g_2, \bar q^+) \;=\; \frac{(-ig^2)}{S_{q\bar q}}\langle q-|\gamma_\mu|\bar q-\rangle\,((p_1-p_2)^\mu\epsilon_1\cdot\epsilon_2 \;+\; 2\epsilon_1\cdot p_2\epsilon_2^\mu \;-\; 2\epsilon_2\cdot p_1\epsilon_1^\mu)$$

$$+\;\frac{(-ig^2)}{S_{qp_1}}\langle q-|\,\hat\epsilon(p_1)(\hat q+\hat p_1)\hat\epsilon(p2)\,|\bar q-\rangle. \tag{2.25}$$

For both gluons with negative or positive helicity it is trivial to show that this subamplitude vanishes, by choosing the same reference momentum k for both gluons,

$$m(q^-, g_1^+, g_2^+, \bar q^+) \;=\; m(q^-, g_1^-, g_2^-, \bar q^+) = 0 \tag{2.26}$$

In the case that the gluons have opposite helicity then

$$m(q^-, g_1^-, g_2^+, \bar q^+) \;=\; ig^2\frac{\langle q1\rangle^3\langle\bar q1\rangle}{\langle q1\rangle\langle 12\rangle\langle 2\bar q\rangle\langle\bar q q\rangle} \tag{2.27}$$

and

$$m(q^-, g_1^-, g_2^-, \bar{q}^+) = ig^2 \frac{\langle q2\rangle^3 \langle \bar{q}2\rangle}{\langle q1\rangle\langle 12\rangle\langle 2\bar{q}\rangle\langle \bar{q}q\rangle}. \tag{2.28}$$

which is most easily shown by choosing the reference momentum for gluon one to be the momentum of the second gluon and vice versa; with this choice the s-channel contribution vanishes.

In general the amplitudes with all particles or all but one particle having the same helicity vanishes at tree level. Also the quark and the antiquark must have opposite helicity or otherwise the amplitude vanishes from chirality conservation in massless QCD. The amplitudes with one gluon the same helicity as the quark or antiquark and all other gluons having the opposite helicity have simple expressions:

$$\begin{aligned} m_{\bar{q}q}(\bar{q}^+, q^-, g_1^+, \ldots, g_I^-, \ldots, g_n^+) &= ig^n \frac{\langle \bar{q}I\rangle\langle qI\rangle^3}{\langle \bar{q}q\rangle\langle q1\rangle\langle 12\rangle \cdots \langle n\bar{q}\rangle}. \\ m_{\bar{q}q}(\bar{q}^-, q^+, g_1^+, \ldots, g_I^-, \ldots, g_n^+) &= ig^n \frac{\langle \bar{q}I\rangle^3\langle qI\rangle}{\langle \bar{q}q\rangle\langle q1\rangle\langle 12\rangle \cdots \langle n\bar{q}\rangle}. \end{aligned} \tag{2.29}$$

Other amplitudes are more complex, e.g. the general form of the quark-antiquark four gluon amplitude has the following pole structure as dictated by duality[13]:

$$\begin{aligned} m_{\bar{q}q}(\bar{q}, q, g_1, g_2, g_3, g_4) &= ig^4 \left[\frac{P_1}{t_{\bar{q}q1} S_{\bar{q}q} S_{q1} S_{23} S_{34}} + \frac{P_2}{t_{q12} S_{q1} S_{12} S_{34} S_{4\bar{q}}} \right. \\ &\left. + \frac{P_3}{t_{123} S_{12} S_{23} S_{4\bar{q}} S_{\bar{q}q}} + \frac{P_4}{S_{\bar{q}q} S_{q1} S_{12} S_{23} S_{34} S_{4\bar{q}}} \right]. \end{aligned} \tag{2.30}$$

The numerators P_i are complicated and I refer you to reference [10] for explicit expressions for these quantities. The sub-amplitudes defined by equation (2.22) have similar properties to the purely gluonic ones in the soft, collinear and multi-particle pole limits, see Section 3.

To construct the QED results from the non-abelian amplitudes all that is needed is to replace $\lambda_{\alpha\bar{\alpha}}$. in equation (2.22) by $\delta_{\alpha\bar{\alpha}}$, for details see reference [11]. For example, one of the helicity amplitudes involving an electron, positron and n photons can be written as

$$\begin{aligned} \mathcal{M}(\bar{e}^+, e^-; \gamma_1^+, \ldots \gamma_I^-, \ldots, \gamma_n^+) &= ie_0^n \frac{\langle \bar{e}I\rangle\langle eI\rangle^3}{\langle \bar{e}e\rangle} \sum_{\{1,\ldots,n\}} \frac{1}{\langle e1\rangle\langle 12\rangle \cdots \langle n\bar{e}\rangle} \\ &= ie_0^n \frac{\langle \bar{e}I\rangle\langle eI\rangle^3}{\langle \bar{e}e\rangle^2} \prod_j \frac{\langle \bar{e}e\rangle}{\langle ej\rangle\langle j\bar{e}\rangle}. \end{aligned} \tag{2.31}$$

In this example one can see that for the abelian amplitude the photons are emitted independently of each other whereas for the non-abelian amplitude there is a correlation between the emitted gluons.

To extend this quark dual basis to more than one quark antiquark pair I refer you to the paper by Mangano [11], and here I briefly sketch the color basis. For two quark-antiquark pairs of different flavors with colors α, α' and $\bar{\alpha}$, $\bar{\alpha}'$ the dual color expansion is

$$\begin{aligned}\mathcal{M}_{\bar{q}q\bar{q}'q'ng} &= \sum_{\{\sigma,\tau\}} (\prod_{\sigma} \lambda)_{\alpha\bar{\alpha}'}(\prod_{\tau} \lambda)_{\alpha'\bar{\alpha}}\; m^{\neq}_{\bar{q}q\bar{q}'q'}(\sigma,\tau) \\ &\qquad + \frac{1}{N}\,(\prod_{\sigma} \lambda)_{\alpha\bar{\alpha}}(\prod_{\tau} \lambda)_{\alpha'\bar{\alpha}'}\; m^{0}_{\bar{q}q\bar{q}'q'}(\sigma,\tau) \end{aligned} \qquad (2.32)$$

where the sum is over all partitions, $\{\sigma,\tau\}$, of all permutations of the n gluons. The first term is the contribution in which the color flow connects α to $\bar{\alpha}'$ and α' to $\bar{\alpha}$ whereas the second term comes from the color flow connecting α to $\bar{\alpha}$ and α' to $\bar{\alpha}'$. For two quark pairs of the same flavor one must add $\mathcal{M}_{\bar{q}q'\bar{q}'qng}$ to $\mathcal{M}_{\bar{q}q\bar{q}'q'ng}$. Similar factorization properties to $m_{\bar{q}q}$ also hold for $m^{\neq}_{\bar{q}q\bar{q}'q'}$ and $m^{0}_{\bar{q}q\bar{q}'q'}$.

2.3 Pure Gluon Color Decomposition

Consider an $SU(N)$ Yang-Mills theory, then at *tree level* in perturbation theory, any vector particle scattering amplitude, with colors $a_1, a_2 \ldots a_n$, external momenta $p_1, p_2 \ldots p_n$ and helicities $\epsilon_1, \epsilon_2 \ldots \epsilon_n$, can be written as

$$\mathcal{M}_{ng} = \sum_{perm'} tr\,(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n})\, m(p_1,\epsilon_1;p_2,\epsilon_2;\cdots;p_n,\epsilon_n), \qquad (2.33)$$

where the sum, $perm'$, is over all $(n-1)!$ *non-cyclic* permutations of $1, 2, \ldots, n$ and the λ's are the matrices of the symmetry group in the fundamental representation. This expansion is known as the dual expansion because of the invariance of the sub-amplitudes under cyclic permutations[12].

The proof that one can always make this expansion is very simple using the identities $[\lambda^a, \lambda^b] = i\sqrt{2} f_{abc}\lambda^c$ and $tr(\lambda^a\lambda^b) = \delta^{ab}$. In any tree level Feynman diagram, replace the color structure function at some vertex using

$$f_{abc} = -(i/\sqrt{2})\; tr(\lambda^a\lambda^b\lambda^c \;-\; \lambda^c\lambda^b\lambda^a). \qquad (2.34)$$

Now each leg attached to this vertex has a λ matrix associated with it. At the other end of each of these legs there is either another vertex or this is an external leg. If there is another vertex, use the λ associated with this internal leg to write the structure function of this vertex $f_{cde}\,\lambda^c$ as $-i\,[\lambda^d, \lambda^e]/\sqrt{2}$. Continue this processes until all vertices have been treated in this manner. Then this Feynman diagram has been placed in the form of eqn(2.33). Repeating this procedure for all Feynman diagrams for a given process completes the proof.

The sub-amplitudes $m(1,2,\ldots,n) \equiv m(p_1,\epsilon_1;p_2,\epsilon_2;\ldots p_n,\epsilon_n)$ of eqn(2.33) satisfy a number of important properties and relationships.

(1) $m(1,2,\ldots,n)$ is gauge invariant.
(2) $m(1,2,\ldots,n)$ is invariant under cyclic permutations of $1,2,\ldots,n$
(3) $m(n,n-1,\ldots,1) = (-1)^n m(1,2,\ldots,n)$
(4) The Dual Ward Identity:

$$\begin{aligned} m(1,2,3,\ldots,n) \;+\; m(2,1,3,\ldots,n) \;+\; m(2,3,1,\ldots,n) \\ +\;\cdots\;+\; m(2,3,\ldots,1,n) \;=\; 0 \end{aligned} \tag{2.35}$$

(5) Factorization of $m(1,2,\cdots,n)$ in the soft, collinear and multi-gluon pole limits.
(6) Incoherence to leading order in number of colors:

$$\sum_{colors} |\mathcal{M}_{ng}|^2 \;=\; N^{n-2}(N^2-1)\sum_{perm'}\left\{|m(1,2,\cdots,n)|^2 + \mathcal{O}(N^{-2})\right\}. \tag{2.36}$$

This set of properties for the sub-amplitudes, we will refer to as duality and the expansion in terms of these dual sub-amplitudes the dual expansion. Properties (1) and (2) can be seen directly from the properties of linear independence, for arbitrary N, and invariance under cyclic permutations of $tr\,(\lambda^1\lambda^2\ldots\lambda^n)$. Whereas (3) and (4) follow by studying the sum of Feynman diagrams which contribute to each sub-amplitude. The sum of Feynman diagrams which make the Dual Ward Identity is such that each diagram is paired with another with opposite sign so that the combination contained in eqn(2.35) trivially vanishes. Property (5) will be discussed in great detail in section 3 and the incoherence to leading order in the number of colors (6) follows from the color algebra of the $SU(N)$ gauge group.

To the string theorist this expansion and the duality properties (1) to (6), see [13], are quite familar since the string amplitude, in the zero slope limit, reproduces the Yang-Mills amplitude on mass shell [14]. Each sub-amplitude is then represented by the zero slope limit of a string diagram, and the sub-amplitude could be obtained by using the usual Koba-Nielsen formula [15]. The traces of λ matrices are just the Chan-Paton factors[16]. For the string amplitude the properties (1) through (6) are satisfied even before the zero slope limit is taken. Also from the string diagrams it is simple to see which Feynman diagrams contribute to a given sub-amplitude, e.g. Fig. 3. The coefficients for the contributing diagrams are obtained by the procedure developed earlier in this section for re-writing the color factors. The relationship between the string diagram and our dual sub-amplitudes suggests that a Yang-Mills amplitude expressed in terms of these dual sub-amplitudes will assume a particularly simple form.

The gauge invariance and properties under cyclic and reverse permutations allows the calculation of far fewer than the $(n-1)!$ sub-amplitudes that appear in the dual

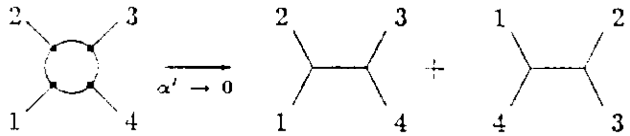

Figure 3: The zero-slope limit of the four gluon string diagram in terms of Feynman diagrams (tri-gluon couplings only).

expansion. In fact the number of sub-amplitudes that are needed is just the number of different orderings of positive and negative helicities around a circle. Of course some of the sub-amplitudes vanish because of the partial helicity conservation of tree level Yang-Mills and others are simply related to one another through the properties (2) through (4).

For four gluon scattering only the helicity conserving amplitudes are non zero. There are many ways to see this, here we will give the most direct one. Each term in the four gluon amplitude contains at least one $\epsilon \cdot \epsilon'$ factor. By choosing appropriate reference momenta you can make all $\epsilon \cdot \epsilon'$ factors zero in all helicity configurations except those that are helicity conserving. Using the convention that all particles are labelled with their helicities and momenta as if they were outgoing, i.e. the incoming particles have negative energies, the helicity conserving sub-amplitude $(1^+, 2^-, 3^-, 4^+)$ is given by

$$\begin{aligned} m(1_3^+, 2_4^-, 3_4^-, 4_3^+) &= -2ig^2 \frac{\epsilon(p_1, p_3) \cdot \epsilon(p_2, p_4)\ \epsilon(p_3, p_4) \cdot p_2\ \epsilon(p_4, p_3) \cdot p_1}{S_{23}} \\ &= ig^2 \frac{\langle 23 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \end{aligned} \tag{2.37}$$

where the gluons are labelled with superscripts which are the helicities and subscripts which are the reference momentum. For this helicity configuration and this choice of reference momenta the only non-zero $\epsilon \cdot \epsilon'$ is $\epsilon_1 \cdot \epsilon_2$.

In general the four gluon scattering subamplitudes are given by

$$\begin{aligned} m_{2+2-}(1,2,3,4) &= -ig^2 \frac{\langle IJ \rangle^2 [KL]^2}{S_{12}\ S_{23}} \\ &= ig^2 \frac{\langle IJ \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \end{aligned} \tag{2.38}$$

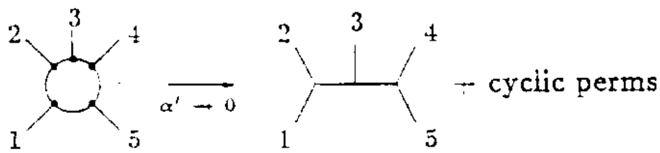

Figure 4: The zero-slope limit of the five gluon string diagram in terms of Feynman diagrams (tri-gluon couplings only).

The momenta I and J (K and L) in the numerator are the momenta of the negative (positive) helicity gluons independent of their ordering in the sub-amplitude, whereas the order of the spinor products in the denominator is only determined by the order of the momenta in the sub-amplitude. Using the properties of the spinor product is simple to demonstrate that eqn(2.38) satisfies the four particle Dual Ward Identity (2.35).

In squaring the four gluon amplitude and summing over colors the $\mathcal{O}(N^{-2})$ terms in eqn(2.36) can be shown to vanish by using only the general properties, especially the Dual Ward Identity, of the sub-amplitudes. Therefore,

$$\sum_{colors} |\mathcal{M}_{4g}|^2 = N^2(N^2-1) \sum_{perm'} |m(1,2,3,4)|^2, \tag{2.39}$$

and the square of each sub-amplitude is very simple because the spinor product is the square root of twice the dot product. The final result is the standard four gluon matrix element squared.

$$\sum_{hel.} \sum_{colors} |\mathcal{M}_{4g}|^2 = N^2(N^2-1)\ g^4 \left(\sum_{i>j} S_{ij}^4\right) \sum_{perm'} \frac{1}{S_{12}S_{23}S_{34}S_{41}}. \tag{2.40}$$

Here we have not averaged over incoming helicities or colors.

For five gluon scattering only those Feynman diagrams, or part there of, with color structure the same as the diagrams of Fig. 4 contribute to the $m(1,2,3,4,5)$ sub-amplitude.

Again, it is a straight forward, simple calculation [12] to show that the only nonzero sub-amplitudes have either two or three negative helicity gluons and that the three positive - two negative helicity sub-amplitude is given by

$$m_{3+2-}(1,2,3,4,5) = ig^3 \frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}. \tag{2.41}$$

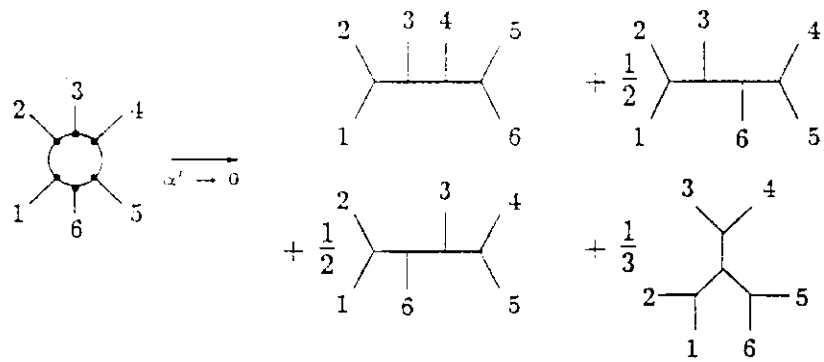

Figure 5: The zero-slope limit of the six gluon string diagram in terms of Feynman diagrams (tri-gluon couplings only).

Where I and J are again the momenta of the negative helicity gluons and the denominator ordering is determined by the order of the momenta in the sub-amplitude. The two positive - three negative helicity amplitude is obtained from this last equation by complex conjugation. By using the Fierz properties of the spinor product it is easy to demonstrate that eqn(2.41) satisfies the five particle Dual Ward Identity, eqn(2.35).

Again, the general properties of the sub-amplitude can be used to show that the $\mathcal{O}(N^{-2})$ terms in eqn(2.36) vanish for the five gluon process giving the following standard result that

$$\sum_{hel.}\sum_{colors} |\mathcal{M}_{5g}|^2 = 2\ N^3(N^2-1)\ g^6\ \left(\sum_{i>j} S_{ij}^4\right) \sum_{perm'} \frac{1}{S_{12}S_{23}S_{34}S_{45}S_{51}}. \quad (2.42)$$

Here we have not averaged over incoming helicities or colors.

For the six gluon process only those Feynman diagrams, or part there of, with the same color structure as the diagrams of Fig. 5 contribute to the $m(1,2,3,4,5,6)$ sub-amplitude. Then, by using the appropriate reference momenta for the polarization vectors it is easy to see that the only non-zero sub-amplitudes are those with four positive - two negative, two positive - four negative and three positive - three negative helicities. After a lengthy calculation we have obtained the following expressions for the six gluon sub-amplitudes.

The sub-amplitudes for the four positive - two negative helicity processes are a straight forward generalization of the four and five-gluon sub-amplitudes:

$$m_{4+2-}(1,2,3,4,5,6) \;=\; ig^4 \frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}. \tag{2.43}$$

Again, I and J represent the momenta of the negative helicity gluons. Different permutations can be obtained as before by keeping fixed the numerator and permuting the momenta in the denominator. The two positive - four negative helicity sub-amplitude is obtained from eqn(2.43) by complex conjugation.

The three positive - three negative helicity sub-amplitudes are not as simple. To exhibit the factorization on the three particle channels these sub-amplitudes are

$$\begin{aligned} m_{3+3-}(1,2,3,4,5,6) \;&=\; ig^4 \left[\frac{\alpha^2}{t_{123}S_{12}S_{23}S_{45}S_{56}} + \frac{\beta^2}{t_{234}S_{23}S_{34}S_{56}S_{61}} \right. \\ &+ \left. \frac{\gamma^2}{t_{345}S_{34}S_{45}S_{61}S_{12}} + \frac{t_{123}\beta\gamma \;+\; t_{234}\gamma\alpha \;+\; t_{345}\alpha\beta}{S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}} \right] \end{aligned} \tag{2.44}$$

where the $t_{ijk} \equiv (p_i + p_j + p_k)^2 \;=\; S_{ij} \;+\; S_{jk} \;+\; S_{ki}$. The coefficients α, β and γ for the three distinct orderings of the helicities are given in Table I. With this representation it is a simple exercise to show that these sub-amplitudes factorize on the three particle pole into a product of two four particle sub-amplitudes, eqn(2.38), times the three particle propagator.

<u>Table I</u>

Coefficients for the m_{3+3-} Sub-amplitudes:
where $\langle I|K|J\rangle \equiv \langle I+|K\cdot\gamma|J+\rangle$, which is linear in K
and if $K^2 = 0$ is given by $[IK]\langle KJ\rangle$.

	$1^+2^+3^+4^-5^-6^-$ $X = 1 + 2 + 3$	$1^+2^+3^-4^+5^-6^-$ $Y = 1 + 2 + 4$	$1^+2^-3^+4^-5^+6^-$ $Z = 1 + 3 + 5$
α	0	$-[12]\langle 56\rangle\langle 4\|Y\|3\rangle$	$[13]\langle 46\rangle\langle 5\|Z\|2\rangle$
β	$[23]\langle 56\rangle\langle 1\|X\|4\rangle$	$[24]\langle 56\rangle\langle 1\|Y\|3\rangle$	$[51]\langle 24\rangle\langle 3\|Z\|6\rangle$
γ	$[12]\langle 45\rangle\langle 3\|X\|6\rangle$	$[12]\langle 35\rangle\langle 4\|Y\|6\rangle$	$[35]\langle 62\rangle\langle 1\|Z\|4\rangle$

The six gluon sub-amplitudes satisfy the three distinct Dual Ward Identities obtained from the following equation

$$\begin{aligned} m(1,2,3,4,5,6) \;+\; m(2,1,3,4,5,6) \;+\; m(2,3,1,4,5,6) \\ +\; m(2,3,4,1,5,6) \;+\; m(2,3,4,5,1,6) \;=\; 0 \end{aligned} \tag{2.45}$$

using the helicity ordering of the first term as either $m(1+,2+,3+,4+,5-,6-)$, $m(1+,2+,3+,4-,5-,6-)$ or $m(1+,2-,3+,4-,5+,6-)$. These three Identities are extremely powerful and relate sub-amplitudes with different orderings of the helicities.

Given the simplicity of the sub-amplitudes with two negative helicities and all the others positive, equations (2.38), (2.41) and (2.43), it is obvious that the generalization to arbitrary n is

$$m_{(n-2)+2-}(1,2,\ldots,n) \quad = \quad ig^{n-2}\,\frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{2.46}$$

where once again I and J are the momenta of the negative helicity gluons. Apart from this being the natural square root of the expression given by Parke and Taylor [17], it also ***satisfies*** the Dual Ward Identity for arbitrary n.

The complete square of the six-gluon amplitude, including the non-leading color terms is

$$\begin{aligned}
&\sum_{colors} |\mathcal{M}_{6g}|^2 \; = \\
N^4(N^2-1) \quad \textstyle\sum_{perm'} \quad &|\, m(1,2,3,4,5,6)\,|^2 \\
&+\frac{1}{N^2}\Big(m^*(1,2,3,4,5,6)\big[m(1,3,5,2,6,4) \\
&\qquad + m(1,3,6,4,2,5) \;+\; m(1,4,2,6,3,5)\big] + c.c\Big).
\end{aligned} \tag{2.47}$$

Note that the sub-amplitudes add incoherently to leading order in the number of colors and the simplicity of the non-leading color terms is achieved by the properties of the sub-amplitudes, especially the Dual Ward Identity equation (2.35). This result together with the expressions for the sub-amplitudes, eqn(2.43) and (2.44), can be used to calculate the matrix element squared by evaluating the sub-amplitudes as complex numbers. Owing to the simplicity of the sub-amplitudes and the simplicity of the leading and non-leading terms in the number of colors this method of calculation is appreciable faster than previous numerical algorithms.

The ordering of the gluons in the non-leading color terms is of particular import. These terms are the only possible ones which have no two or three particle propagators in common with the original ordering $(1,2,3,4,5,6)$ and as such are less singular in the collinear limit than the leading part in N. In fact the non-leading color terms are finite in the collinear limit so that in this limit they are completely irrelevant compared to the leading color terms. Also by comparing numerically the leading to non-leading pieces for N$=3$, the non-leading terms contribute in general only a few percent to the total cross-section. This result is even true in the soft gluon limit. Therefore the non-leading terms can be ignored given that this calculation is only to tree level, and the other uncertainties in any Monte Carlo application are much larger than this uncertain. The smallness of the non-leading color terms and the fact that the leading color terms are just the squares of the simple sub-amplitudes implies that the square of this matrix element is easy to obtain.

3 Factorization Properties

The most important and remarkable properties of the Yang-Mills dual sub-amplitudes are their factorization properties, whose origin can be traced back to the string picture. In this section we give the factorization properties of the gluon sub-amplitudes in

(1) the soft gluon limit,
(2) when two gluons become collinear and
(3) when three gluons add to form an on mass-shell gluon
i.e. on the three gluon pole.

For arbitrary n-gluon scattering these factorization properties of the sub-amplitudes will extend up to factorization on the $[n/2]$-gluon poles.

First, we consider the soft gluon limit. Consider the sub-amplitudes when gluon 1 has an energy which is small compared to all the other energies in the process. Then the gluon sub-amplitudes must satisfy

$$m(1^+, 2\ldots, n) \quad \xrightarrow{1^+\ soft} \quad \left\{\frac{g\ \langle n\ 2\rangle}{\langle n\ 1\rangle\langle 1\ 2\rangle}\right\} \quad m(2, 3\ldots, n) \tag{3.1}$$

$$m(1^-, 2\ldots, n) \quad \xrightarrow{1^-\ soft} \quad \left\{\frac{g\ [n\ 2]}{[n\ 1][1\ 2]}\right\} \quad m(2, 3\ldots, n). \tag{3.2}$$

The factors in braces are square roots of the eikonal factor

$$\frac{g^2\ (p_n \cdot p_2)}{(p_n \cdot p_1)\ (p_1 \cdot p_2)}.$$

This soft gluon factorization and the incoherence of these sub-amplitudes to leading order in the number of colors, N, leads to the soft gluon factorization of the full matrix element squared as proposed by Bassetto, Ciafaloni and Marchesini [18],

$$\sum_{colors} |\mathcal{M}_{ng}|^2 \quad \xrightarrow{1\ soft} \quad \sum_{ij} \left(\frac{g^2\ (p_i \cdot p_j)}{(p_i \cdot p_1)\ (p_1 \cdot p_j)}\right) |A_{ij}(2, \cdots, n)|^2. \tag{3.3}$$

In the limit when two gluons become collinear, Altarelli and Parisi [19] demonstrated that the double poles associated with this collinear pair do not appear in the full amplitude squared i.e. there is a cancellation of one power of the propagator of the sum of the two collinear gluons. This cancellation occurs at the amplitude level rather than the square of the amplitude in this dual formulation. Therefore the squared sub-amplitudes diverge no more rapidly than a single power of the propagator for the collinear gluons, this is the Altarelli and Parisi observation. The origin of this behaviour of the dual sub-amplitudes stems from the factorization properties of string amplitudes.

To demonstrate this square root divergence of the sub-amplitudes in the collinear limit, consider the case when the momenta of particles 1 and 2 become parallel. Let $1 \to z\ P$ and $2 \to (1-z)\ P$ with $P^2 = 0$, and z is the momentum fraction of particle 1. Then the sub-amplitudes become

$$m(1^+,2^+,3,\ldots) \xrightarrow{1^+ \parallel 2^+} \left\{ \frac{ig\ [12]}{\sqrt{z(1-z)}} \right\} \frac{-i}{S_{12}}\ m(P^+,3,\ldots) \tag{3.4}$$

$$m(1^+,2^-,3,\ldots) \xrightarrow{1^+ \parallel 2^-} \left\{ \frac{ig\ z^2\langle 12\rangle}{\sqrt{z(1-z)}} \right\} \frac{-i}{S_{12}}\ m(P^+,3,\ldots) \tag{3.5}$$

$$+ \left\{ \frac{ig\ (1-z)^2\ [12]}{\sqrt{z(1-z)}} \right\} \frac{-i}{S_{12}}\ m(P^-,3,\ldots)$$

$$m(1^-,2^-,3,\ldots) \xrightarrow{1^- \parallel 2^-} \left\{ \frac{ig\ \langle 12\rangle}{\sqrt{z(1-z)}} \right\} \frac{-i}{S_{12}}\ m(P^-,3,\ldots). \tag{3.6}$$

Note that either $\langle 12\rangle$ or $[12]$ appears in the numerator of each term. Also, it is useful to interpret the factor in braces as the "three gluon sub-amplitude" in the limit when two gluons become collinear. This three gluon sub-amplitude has the square root suppression of the pole as well as having the square root of the appropriate Altarelli-Parisi gluon-fusion function. From this result and the incoherence of the sub-amplitudes in the square of the matrix element the standard results of Altarelli and Parisi are obtained in a simple manner.

The sub-amplitudes also factorize in the three particle channel; here let $P = 1+2+3$, then as $P^2 \to 0$ it is easy to see that

$$m(1,2,3,4,5,6) \quad \to \quad m(1,2,3,-P)\ \frac{-i}{P^2}\ m(P,4,5,6) \tag{3.7}$$

for the helicity structure three positive and three negative. Since helicity is conserved in the four gluon process, the helicity of the intermediate gluon is determined for this helicity structure and the four positive - two negative helicity sub-amplitude has no three particle poles.

Of course the full matrix element must also factorize. This is trivial in Feynman diagram language but here it is not so obvious because of the way we have added diagrams together. The color factors almost factorizes for an $SU(N)$ gauge group,

$$\begin{aligned} tr\,(\lambda^1\lambda^2\ldots\lambda^n) &= \sum_x tr\,(\lambda^1\ldots\lambda^m\lambda^x)tr\,(\lambda^x\lambda^{m+1}\ldots\lambda^n) \\ &\quad - \frac{1}{N}tr\,(\lambda^1\ldots\lambda^m)tr\,(\lambda^{m+1}\ldots\lambda^n). \end{aligned} \tag{3.8}$$

This "factorization" property of the traces follows from the identity

$$\sum_a \lambda^a_{ij}\ \lambda^a_{kl} \quad = \quad (\delta_{il}\ \delta_{jk} - \frac{1}{N}\delta_{ij}\ \delta_{kl}). \tag{3.9}$$

The $1/N$ term could destroy the full factorization, but it does not. Terms proportional to $1/N$ vanish at the pole because of the Dual Ward Identity for the sub-amplitudes. Therefore, all the gluon amplitudes discussed in this paper satisfy, as expected, the factorization property

$$\mathcal{M}_{n+n'} \ \rightarrow\ \sum \mathcal{M}_{n+1}\ \frac{-i}{P^2}\ \mathcal{M}_{n'+1} \tag{3.10}$$

as $P^2 \rightarrow 0$ for $n, n' \geq 2$. The sum is over the color and helicity of the intermediate state.

From this multi-particle factorization we can understand why the helicity amplitudes $(-\ -\ +\ldots+)$ and $(+\ +\ -\ldots-)$ do not contain any propagators with more than two particles. The residue on a pole with more than two particles in the propagator is a product of two subamplitudes one of which has the following helicity structures $(+\ldots+)$, $(-\ +\ldots+)$, $(+\ -\ldots-)$ or $(-\ldots-)$. All of which have been shown to vanish. Therefore these special helicity structures have only soft and collinear poles.

4 Recursive Relations

The color structure for purely gluonic and processes involving gluons and a quark-antiquark pair defined in previous sections allows for the reorganization of the perturbation theory in a efficient and straight forward manner. The building blocks are color ordered vector and spinorial currents defined with a gluon off mass shell, or a quark or antiquark off mass shell, with all other particles on mass shell. If you have calculated these building blocks for n on mass shell legs then there are recursion relationships, the Berends-Giele recursion relations, ref.[20], which allow you to simply evaluate these currents with $(n+1)$ on mass shell legs. This allows for computer evaluation of processes with a large number of external particles[21]. A detailed and self-contained description of the use of recursive relations in the calculation of multi-parton processes can be found in Giele's thesis [22].

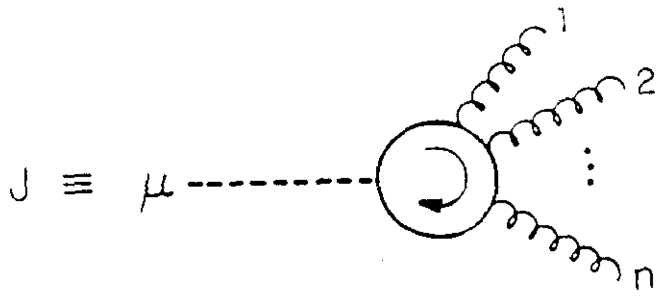

Figure 6: The color ordered gluonic current.

4.1 Color Ordered Gluon Currents

From the set of color truncated Feynman diagrams that make up the subamplitude, $m(1,2,\ldots,n)$, one can form a color ordered gluonic current by replacing the polarization vector of the $n-th$ gluon with the propagator and allowing the momentum of this gluon to be off mass shell but still retain momentum conservation. This color ordered gluonic current will be represented by Fig. 6, where the dotted line represents the gluon which is off mass shell. This current will be written as $J_\mu(1,\ldots,n-1)$ and the subamplitude can be reconstructed from this current by multiplying by the inverse propagator and contracting with a polarization vector and allowing the momentum of this gluon to be on mass shell,

$$m(1,2,\ldots,n) \;=\; \{\epsilon^\mu(p_n)\; i[P(1,n-1)]^2\; J_\mu(1,\ldots,n-1)\}\;|_{P(1,n-1)=-p_n}, \qquad (4.1)$$

where, $P(1,n) \equiv \sum_1^n p_i$.

Of course these currents, J_μ, are not gauge invariant and do depend on the choice of reference momenta chosen for the $(n-1)$ on mass shell gluons. Also they depend on the helicity of the on mass shell gluons. However these color ordered gluonic currents can be used as building blocks for gluonic currents with more external on mass shell legs.

Consider a gluonic current with n on mass shell gluons. Then the off mass shell gluon is attached to the rest of the gluons either through a three or a four point color ordered gluon coupling. At these vertices the other legs are attached to color ordered gluonic currents with fewer than n on mass shell gluons. This can be seen diagrammatically in Fig. 7. Hence, the color ordered gluonic current with n on mass shell gluons can be written in terms of gluonic currents with less than n on mass shell gluons. This is the Berends-Giele recursion relation[20] for gluonic color ordered currents and algebraically it is written as

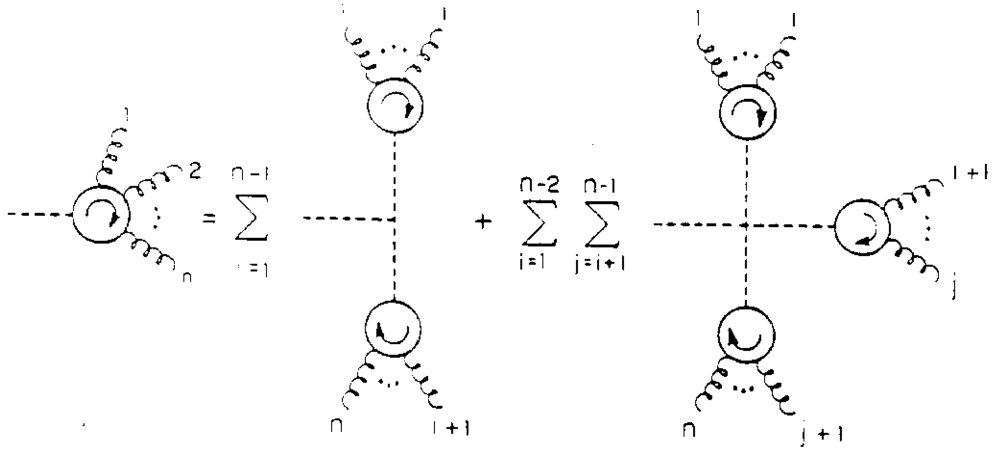

Figure 7: A graphical representation of the Berends-Giele gluonic recursion relation.

$$
\begin{aligned}
J_\mu(1,\ldots,n) \quad &= \frac{-i}{P(1,n)^2}\{\sum_{i=1}^{n-1} V3^{\mu\nu\rho}(P(1,i),P(i+1,n))\ J_\nu(1,\ldots,i)\ J_\rho(i+1,\ldots,n) \\
+ \sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2} \quad & V4^{\mu\nu\rho\sigma}\ \ J_\nu(1,\ldots,i)\ J_\rho(i+1,\ldots,j)\ J_\sigma(j+1,\ldots,n)\} \qquad (4.2)
\end{aligned}
$$

where the color ordered three and four gluon vertices are,

$$
\begin{aligned}
V3^{\mu\nu\rho}(P,Q) \quad &= \quad i\frac{g}{\sqrt{2}}\ (g^{\nu\rho}\ (P-Q)^\mu\ -\ 2g^{\rho\mu}\ Q^\nu\ -\ 2g^{\mu\nu}\ P^\rho), \\
V4^{\mu\nu\rho\sigma} \quad &= \quad i\frac{g^2}{2}\ (2g^{\mu\rho}\ g^{\nu\sigma}\ -\ g^{\mu\nu}\ g^{\rho\sigma}\ -\ g^{\mu\sigma}\ g^{\nu\rho}). \qquad (4.3)
\end{aligned}
$$

The current with one on mass shell gluon is defined as

$$J_\mu(1)\ \equiv\ \epsilon_\mu(p_1). \qquad (4.4)$$

The gluonic currents, $J_\mu(1,\ldots,n)$, satisfy properties that are similar to the gluon subamplitude, $m(1,2,\ldots,n)$.

1. Dual Ward identity:

$$J_\mu(1,2,3,\ldots,n)\ +\ J_\mu(2,1,3,\ldots,n)\cdots\ +\ J_\mu(2,3,\ldots,n,1)\ = 0. \qquad (4.5)$$

2. Reflectivity:

$$J_\mu(1,\ldots,n) \;=\; (-1)^{n+1}\; J_\mu(n,\ldots,1) \tag{4.6}$$

3. $J_\mu(1,\ldots,n)$ is conserved:

$$P(1,n)^\mu\; J_\mu(1,\ldots,n) \;=\; 0 \tag{4.7}$$

There are simple analytical expressions for the color ordered gluonic currents if all the helicities are the same or if one is different from the others. Of course we must define the reference momentum for the gluons. Here the symbol i for the gluons must be expanded to i_k^λ where the i-th gluon has helicity λ and reference light-like momentum k. Then

$$J_\mu(1_k^+, 2_k^+, \ldots, n_k^+) \;=\; g^{n-1} \frac{\langle k-|\;\gamma_\mu\;\hat{P}(1,n)\;|k+\rangle}{\sqrt{2}\;\langle k1\rangle\;\langle 12\rangle\cdots\langle n-1n\rangle\;\langle nk\rangle} \tag{4.8}$$

and

$$J_\mu(1_k^-, 2_k^-, \ldots, n_k^-) \;=\; (-1)^n\; g^{n-1} \frac{\langle k+|\;\gamma_\mu\;\hat{P}(1,n)\;|k-\rangle}{\sqrt{2}\;[k1]\;[12]\cdots[n-1n]\;[nk]}. \tag{4.9}$$

Berends and Giele, ref.[20], give compact expressions for $J_\mu(1^\mp, 2^\pm, \ldots, n^\pm)$ for a given choice of reference momenta.

4.2 Color Ordered Quark Currents

For the subamplitudes involving a quark-antiquark pair and gluons one can define a Quark and Antiquark color ordered spinorial current, see Fig. 8, in a way similar to the gluon currents that were defined in the last section. We will write the Quark current as $\overline{U}(q,1,\ldots,n)$ and the Antiquark current as $V(1,\ldots,n,\bar{q})$.

The quark-antiquark pair plus gluon subamplitudes can be obtained from these currents as follows:

$$\begin{aligned} m(q,1,\ldots,n,\bar{q}) \quad &= \langle q\,|\,(+i)(\hat{\bar{q}} + \hat{P}(1,n))V(1,\ldots,n,\bar{q})\;|_{\bar{q}+P(1,n)=-q} \\ &= \overline{U}(q,1,\ldots,n)(-i)(\hat{q} + \hat{P}(1,n))\,|\bar{q}\rangle\;|_{q+P(1,n)=-\bar{q}} \end{aligned} \tag{4.10}$$

In manner similar to the gluon current, a recursion relation can be written for this color ordered Quark current [20], see Fig. 9,

$$\overline{U}(q,1,\ldots,n) \;=\; \sum_{m=0}^{n-1} \overline{U}(q,1,\ldots,m)\frac{ig}{\sqrt{2}}\gamma^\mu J_\mu(m+1,\ldots,n)\frac{i}{(\hat{q}\;+\;\hat{P}(1,n))} \tag{4.11}$$

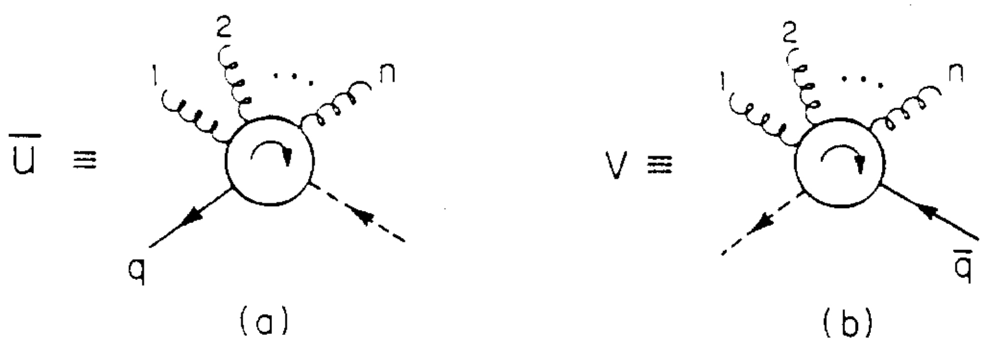

Figure 8: The quark, (a), and antiquark, (b) color ordered spinorial currents.

and for the Anti-quark current[20]

$$V(1,\ldots,n,\overline{q}) = \sum_{m=1}^{n} \frac{-i}{(\hat{\overline{q}} + \hat{P}(1,n))} \frac{ig}{\sqrt{2}} \gamma^\mu J_\mu(1,\ldots,m) V(m+1,\ldots,n,\overline{q}) \tag{4.12}$$

and where the spinor currents for the zero gluon case are defined to be

$$\overline{U}(q) \;\equiv\; \overline{u}(q), \qquad V(\overline{q}) \;\equiv\; v(\overline{q}) \tag{4.13}$$

in Bjorken and Drell notation.

These color ordered spinor currents can be defined for massive or massless quarks. For massive quarks the propagators in the recursion relations Eqs. (4.11,4.12) must be modified by adding the appropriate mass term. For massless quarks these spinor currents carry a chirality such that

$$(1 \pm \gamma_5)\, V(1,\ldots,n,\overline{q}^{\pm}) \;=0, \qquad \overline{U}(q^{\pm},1,\ldots,n)(1 \pm \gamma_5) \;=0. \tag{4.14}$$

Also for the massless case the zero gluon currents are simply

$$\overline{U}(q^{\pm}) \;\equiv\; \langle q \pm |, \qquad V(\overline{q}^{\pm}) \;\equiv\; |\overline{q} \mp \rangle. \tag{4.15}$$

Again there are simple analytic expressions for these color ordered spinor currents when all the gluons have the same helicity as the fermion,

$$\overline{U}(q^+,1_k^+,\ldots,n_k^+) \;=\; -g^n \,\frac{\langle k - | (\hat{q} + \hat{P}(1,n))}{\langle q1\rangle\langle 12\rangle \cdots \langle nk\rangle}, \tag{4.16}$$

$$\overline{U}(q^-,1_k^-,\ldots,n_k^-) \;=\; -(-g)^n \,\frac{\langle k + | (\hat{q} + \hat{P}(1,n))}{[q1][12] \cdots [nk]}, \tag{4.17}$$

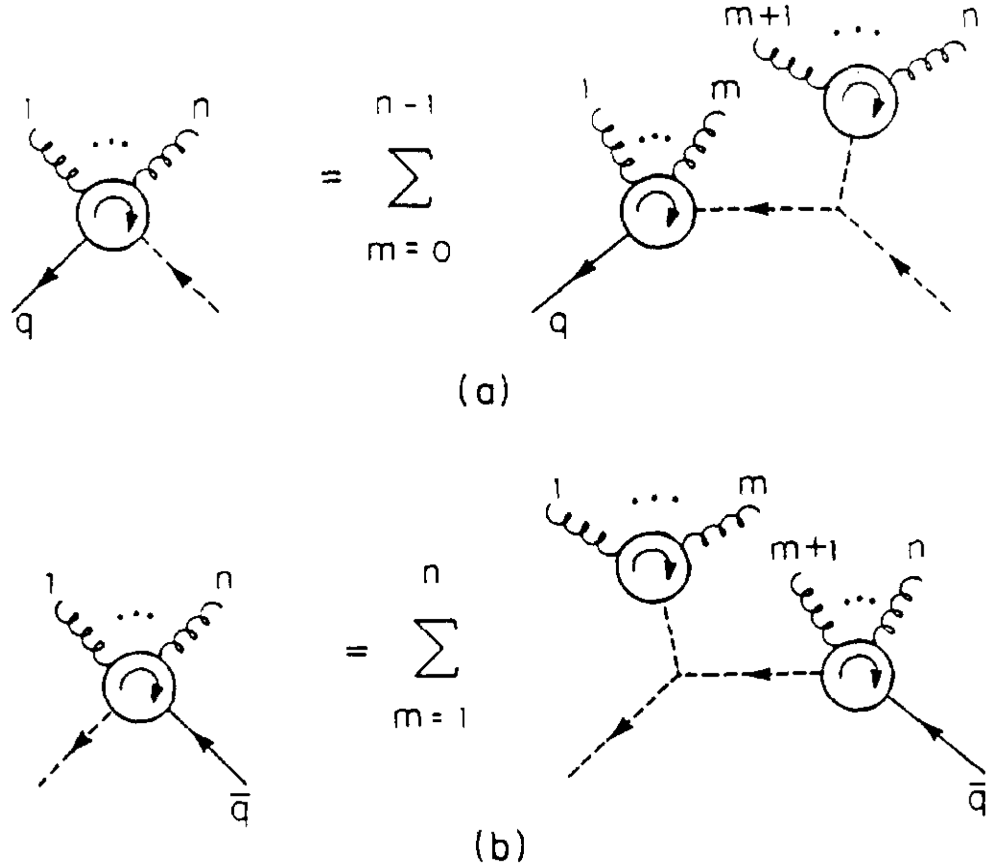

Figure 9: Graphical representations of the Berends-Giele quark, (a), and antiquark, (b), recursion relations.

$$V(1_k^+,\ldots,n_k^+,\overline{q}^+) \;=\; -g^n\,\frac{(\hat{\overline{q}}+\hat{P}(1,n))\,|k+\rangle}{\langle k1\rangle\langle 12\rangle\cdots\langle n\overline{q}\rangle}, \tag{4.18}$$

and

$$V(1_k^-,\ldots,n_k^-,\overline{q}^-) \;=\; -(-g)^n\,\frac{(\hat{\overline{q}}+\hat{P}(1,n))\,|k-\rangle}{[k1][12]\cdots[n\overline{q}]}. \tag{4.19}$$

If there is one gluon with opposite helicity to that of the fermion, the spinorial currents are

$$\overline{U}(q^+,1_k^-) \;=\; -g\,\frac{[qk]\langle q+|}{[q1][1k]}, \tag{4.20}$$

$$\overline{U}(q^-,1_k^+) = g\,\frac{\langle qk\rangle\langle q-|}{\langle q1\rangle\langle 1k\rangle}, \tag{4.21}$$

$$V(1_k^-,\overline{q}^+) = -g\,\frac{|\overline{q}-\rangle[k\overline{q}]}{[k1][1\overline{q}]} \tag{4.22}$$

and

$$V(1_k^+,\overline{q}^-) = g\,\frac{|\overline{q}+\rangle\langle k\overline{q}\rangle}{\langle k1\rangle\langle 1\overline{q}\rangle}. \tag{4.23}$$

Finally, for two gluons with opposite helicity, we have the following spinorial currents,

$$\overline{U}(q^-,1_2^+,2_1^-) = \frac{-g^2\ \langle q2\rangle^2}{\langle q1\rangle S_{12}(q+1+2)^2}\langle 1+|(\hat{q}+\hat{1}+\hat{2}), \tag{4.24}$$

$$\overline{U}(q^-,1_2^-,2_1^+) = \frac{g^2\ [2q]\langle q1\rangle}{[q1]S_{12}(q+1+2)^2}\langle 2+|(\hat{q}+\hat{1}+\hat{2}), \tag{4.25}$$

$$V(1_2^+,2_1^-,\overline{q}^+) = (\hat{1}+\hat{2}+\hat{\overline{q}})\,|2+\rangle\frac{-g^2\ [1\overline{q}]^2}{(1+2+\overline{q})^2 S_{12}[2\overline{q}]} \tag{4.26}$$

and

$$V(1_2^-,2_1^+,\overline{q}^+) = (\hat{1}+\hat{2}+\hat{\overline{q}})\,|1+\rangle\frac{g^2\ [2\overline{q}]\langle\overline{q}1\rangle}{(1+2+\overline{q})^2 S_{12}\langle 2\overline{q}\rangle}. \tag{4.27}$$

A straight forward example using these currents is to calculate the sub-amplitude for $(q^-,1^+,2^-,\overline{q}^+)$ process,

$$\begin{aligned} m(q^-,1^+,2^-,\overline{q}^+) &= \overline{U}(q^-)\ i(\hat{1}+\hat{2}+\hat{\overline{q}})\ V(1_2^+,2_1^-,\overline{q}^+) \\ &= \frac{ig^2\ \langle q2\rangle^3\langle\overline{q}2\rangle}{\langle q1\rangle\langle 12\rangle\langle 2\overline{q}\rangle\langle\overline{q}q\rangle}, \end{aligned} \tag{4.28}$$

which is the previously obtained result. The spinorial currents defined here can be used to derive many of the results of other section, especially the section involving multiple gauge groups.

4.3 The Insertion of a W or Z

A spontaneously broken gauge group does not have a simple generalization of the previous subsections. However, the insertion of one such massive vector particle, W or Z, can be easily incorporated. Consider the scattering of a quark-antiquark n gluons and a W vector boson. Then the amplitude for this process is written as

$$A(q,1,\ldots,n,\bar{q};W) = \sum_P (\lambda^1 \cdots \lambda^n)_{i\bar{j}}\; m(q,1,\ldots,n,\bar{q};W) \tag{4.29}$$

where the subamplitude can be written as

$$m(q,1,\ldots,n,\bar{q};W) = i\; \epsilon^{\mu}_{W} \;\sum_{i=0}^{n} \overline{U}(q^-,1,\ldots,i)\gamma_\mu \frac{(1-\gamma_5)}{2} V(i+1,\ldots,n,\bar{q}^+). \tag{4.30}$$

Here the recursion techniques have been explicitly used.

This expression can be used in one of two ways: either one can square it directly or allow the W boson to decay into another fermion-antifermion pair. If one squares this expression directly the relationship

$$\sum_{poi} \epsilon^{\mu}_{W} \epsilon^{\nu}_{W}{}^{*} \;=\; -g^{\mu\nu} + \frac{W^\mu W^\nu}{M_W^2} \tag{4.31}$$

can be employed.

The other alternative is to replace the polarization of the W vector boson by the amplitude for it to decay into a lepton-antilepton pair. Then Eq. (4.30) is written as

$$\begin{aligned} m(q,1,\ldots,n,\bar{q};L,\overline{L}) \;&=\; -i\; \overline{U}(L^-)\gamma_\mu \frac{(1-\gamma_5)}{2} V(\overline{L}^+) \\ \times \;&\; \frac{(-g_{\mu\nu} + \frac{W_\mu W_\nu}{M_W^2})}{(W^2 - M_W^2 + iM_W\Gamma_W)} \\ \times \;&\; \sum_{i=0}^{n} \overline{U}(q^-,1,\ldots,i)\gamma_\nu \frac{(1-\gamma_5)}{2} V(i+1,\ldots,n,\bar{q}^+) \end{aligned} \tag{4.32}$$

If we use the fact that the charged lepton is effectively massless, compared to M_W, the Fierz rearrangement gives

$$\begin{aligned} m(q,1,\ldots,n,\bar{q};L,\overline{L}) \;&=\; \\ -2i\sum_{i=0}^{n} \;&\; \frac{\overline{U}(q^-,1,\ldots,i)\,|L^+\rangle\;\langle\overline{L}^+|\,V(i+1,\ldots,n,\bar{q}^+)}{(W^2 - M_W^2 + iM_W\Gamma_W)} \end{aligned} \tag{4.33}$$

Using the results from the recursion relation section of this report, the subamplitude for the process $q\bar{q} \rightarrow \; W \; \rightarrow L\overline{L}$ is

$$\begin{aligned} m_W(q^-,\bar{q}^+;L^-,\overline{L}^+) \;&=\; \frac{-2i\;[\overline{L}\;\bar{q}]\langle qL\rangle}{(W^2 - M_W^2 + iM_W\Gamma_W)} \\ &=\; \frac{2i\;\langle qL\rangle^2\;[\overline{L}L]}{\langle q\bar{q}\rangle\;(W^2 - M_W^2 + iM_W\Gamma_W)} \\ &=\; \frac{2i\;[\bar{q}\;\overline{L}]^2\;\langle\overline{L}L\rangle}{[q\bar{q}]\;(W^2 - M_W^2 + iM_W\Gamma_W)}. \end{aligned} \tag{4.34}$$

Adding n gluons with the same helicity to this process, gives

$$m_W(q^-, g_1^+, \cdots, g_n^+, \overline{q}^+; L^-, \overline{L}^+) = \frac{2i\ \langle qL\rangle^2\ [\overline{L}L]}{\langle q1\rangle\langle 12\rangle\cdots\langle n\overline{q}\rangle\ (W^2 - M_W^2 + iM_W\Gamma_W)}, \tag{4.35}$$

$$m_W(q^-, g_1^-, \cdots, g_n^-, \overline{q}^+; L^-, \overline{L}^+) = \frac{(-1)^n\ 2i\ [\overline{q}\ \overline{L}]^2\ \langle \overline{L}L\rangle}{[q1][12]\cdots[n\overline{q}]\ (W^2 - M_W^2 + iM_W\Gamma_W)}. \tag{4.36}$$

If we add two gluons of opposite helicity, then the sub-amplitudes are

$$\begin{aligned} m_W(q^-, g_1^-, g_2^-, \overline{q}^+; L^-, \overline{L}^+) &= \frac{-2i}{(W^2 - M_W^2 + iM_W\Gamma_W)} \\ \Big(\frac{\langle qL\rangle\langle \overline{L}+|\hat{q}+\hat{W}|2+\rangle[1\overline{q}]^2}{S_{qW}\ S_{12}\ [2\overline{q}]} &- \frac{\langle q2\rangle\langle qL\rangle[\overline{L}\ \overline{q}][1\overline{q}]}{\langle q1\rangle\ S_{12}\ [2\overline{q}]} \\ &+ \frac{\langle q2\rangle^2\langle 1+|\hat{\overline{q}}+\hat{W}|L+\rangle[\overline{L}\ \overline{q}]}{\langle q1\rangle\ S_{12}S_{\overline{q}W}}\Big) \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} m_W(q^-, g_1^-, g_2^+, \overline{q}^+; L^-, \overline{L}^+) &= \frac{2i}{(W^2 - M_W^2 + iM_W\Gamma_W)} \\ \Big(\frac{\langle qL\rangle\ \langle \overline{L}+|\hat{q}+\hat{W}|1+\rangle\ [2\overline{q}]\ \langle 1\overline{q}\rangle}{S_{qW}\ S_{12}\ \langle 2\overline{q}\rangle} &+ \frac{\langle 2+|\hat{\overline{q}}+\hat{W}|L+\rangle\ \langle \overline{L}+|\hat{q}+\hat{W}|1+\rangle}{[q1]\ S_{12}\ \langle 2\overline{q}\rangle} \\ &+ \frac{[q2]\ \langle q1\rangle\ \langle 2+|\hat{\overline{q}}+\hat{W}|L+\rangle[\overline{L}\ \overline{q}]}{[q1]\ S_{12}\ S_{\overline{q}W}}\Big). \end{aligned} \tag{4.38}$$

These expressions reproduce well known results.

The previous discussion can be extended to include the Z boson by decomposing the coupling of the Z to the quarks and leptons into its left and right handed parts and then proceeding as with the W boson. For a complete discussion of the calculation for these processes, including the complete results for processes including a W boson plus five partons see Berends, Giele and Kuijf, ref.[23]. These results agree with those independently obtained by Hagiwara and Zeppenfeld, ref.[24].

5 Approximate Matrix Elements

The techniques described in the previous Sections provide very powerful tools to calculate the matrix elements of very complex processes. As an example, the Berends and Giele recursive relations were recently used for the calculation of 8-gluon scattering [25]. The resulting expressions, however, prove very slow to evaluate numerically because of their complexity, thus making it almost impossible to generate a number of events large enough to perform relevant physics studies.

These considerations, and the importance of having fast event generators to simulate multi-jet processes at high-energy hadron colliders, where these processes will provide important backgrounds to many possible new physics signals, justify the study of approximate expressions which describe sufficiently well the exact matrix elements throughout phase-space and at the same time are simple enough to allow very fast simulations.

Kunszt and Stirling [26] and Maxwell [27] were the first to realize that the Parke and Taylor amplitudes, Equation (2.46), can be properly fudged in a systematic way so as to reproduce the full sum over all the allowed helicity amplitudes for gluonic processes. This idea was later generalized to other processes in which at least one set of helicity amplitudes is known in both hadronic [27, 28, 29] and e^+e^- [30] multi-jet production. In this Section we will describe these various approximation schemes, referring the reader to the original literature for numerical comparisons between them.

5.1 The Kunszt and Stirling Approximation

We will start from the simplest scheme, namely that of Kunszt and Stirling (KS, see Ref.[26]). It amounts to assuming that all of the helicity amplitudes have 'on average' the same value, and therefore the full amplitude can just be obtained by multiplying the Parke and Taylor (PT) expressions by a proper weight, representing the ratio between the number of non-zero helicity configurations and the number of the Maximum Helicity Violating (MHV) configurations whose matrix-elements are described by the PT formula.

This approximation becomes particularly simple when neglecting sub-leading terms in $1/N$. This is justified because the sub-leading terms have softer collinear singularities than the leading ones, and therefore do not contribute substantially to the numerical value of the matrix elements. In particular, for $n = 6$ the sub-leading terms are finite [12], and only contribute of the order of few percent to the full square.

For an n-gluon process the number of MHV amplitudes is $n(n-1)$ if $n > 4$ and $n(n-1)/2$ if $n = 4$. The total number of non-zero helicity amplitudes is instead $2^n - 2(n+1)$. For $n = 4, 5$ these multiplicities coincide, and the PT formula describes the exact results, as is well known. For n larger than 5, the KS approximation gives:

$$d\sigma_{KS}^{gg\to g\ldots g} = \frac{2^n - 2(n+1)}{n(n-1)}\, d\sigma_{PT}^{gg\to g\ldots g} \tag{5.1}$$

For $n = 6, 7$, for example, the fudge factor is 5/3 and 8/3, respectively.

To describe processes with initial state quarks, KS suggest the use of the so called effective structure function approximation, which gives a good description of the two-to-two QCD processes. According to this approximation in most of the relevant phase-space the differential cross-sections for processes initiated by gg, by qg and by qq or $q\bar{q}$ stand in a constant ratio:

$$\mathrm{d}\sigma_{gg} \;:\; \mathrm{d}\sigma_{gq} \;:\; \mathrm{d}\sigma_{qq} \;=\; 1 \;:\; 4/9 \;:\; (4/9)^2. \tag{5.2}$$

In this way the full differential cross-section, weighted by the appropriate structure functions, reads:

$$\mathrm{d}\sigma_{tot} = F(x_1)F(x_2)\mathrm{d}\sigma_{gg}, \tag{5.3}$$

$$F(x) = g(x) + 4/9\,(q(x) + \bar{q}(x)), \tag{5.4}$$

$g(x)$ and $q(x)$ being the gluon and quark structure functions. For $d\sigma_{gg}$, finally, one takes Eq.(5.1).

The KS approximation scheme tends to overestimate the exact results and the effective structure function approximation is less and less accurate for an increasing number of partons in the final state; nevertheless the KS approximation is an extremely useful tool for simple but significant estimates of multi-jet rates and distributions. For comparisons of this scheme with exact calculations, see for example References [26, 28, 29, 21].

5.2 The Infrared Reduction Technique

It is well known that in the limit in which two partons (say i and j) become collinear, a given process can be described in the Weisszäker-Williams (W-W) approximation:

$$d\sigma^{(n)} = \frac{1}{2(p_i p_j)}\, f(z)\, d\sigma^{(n-1)} \tag{5.5}$$

where $f(z)$ is an appropriate function of the fraction of momentum carried by one of the two partons becoming collinear, and $d\sigma^{(n-1)}$ is the partonic cross-section for the effective $(n-1)$-particle process in which the two collinear partons are replaced by the single one into which they merge. On the pole the W-W approximation is nothing but the factorization of the amplitude, discussed in various occasions in the previous Sections. The functions $f(z)$, in the case of a QCD process, are just the Altarelli-Parisi (AP) [19] splitting functions.

The *infrared reduction* technique introduced by Maxwell [27] improves the W-W approximation by using the exact matrix elements for some simple helicity configurations, and derives the other helicity configurations by approximating their relative weights at the closest collinear pole.

Next to a collinear pole (say $p_1 \cdot p_2 \to 0$) each of the non-vanishing helicity amplitudes will factorize in the following way:

$$d\sigma_h^{(n)} = \frac{1}{2(p_1 p_2)} \sum_{h'} f_{h'}(z)\, d\sigma_{h'}^{(n-1)} + \text{finite} \tag{5.6}$$

where h' are the various helicity configurations which can contribute to the factorization, and $f_{h'}(z)$ are the corresponding polarized AP splitting functions, depending on the variable $z = E_1/(E_1 + E_2)$. For the time being we will restrict our attention to gluon scattering. For the full process, factorization is described by Equation (5.5), with $f(z)$ given by:

$$f(z) = g^2 N \frac{1 + z^4 + (1-z)^4}{z(1-z)} \tag{5.7}$$

If we just sum over the PT amplitudes, instead, we obtain:

$$d\sigma_{PT}^{(n)} = \frac{1}{2(p_1 p_2)} f_{PT}(z, s_{ij})\, d\sigma_{PT}^{(n-1)} \tag{5.8}$$

where $d\sigma_{PT}^{(n)}$ is the sum over all the MHV amplitudes, and $f_{PT}(z, s_{ij})$ is given by:

$$f_{PT}(z, s_{ij}) = g^2 N \frac{R + z^4 + (1-z)^4}{z(1-z)} \tag{5.9}$$

$$R = \frac{\sum_{i>j} s_{ij}^4}{\sum_i s_{Pi}^4}, \tag{5.10}$$

the indices i and j being different from the collinear particles, and P being the sum of the collinear momenta.

Equation (5.8) can also be rewritten in the following fashion:

$$d\sigma_{PT}^{(n)} = \chi^{-1} \frac{1}{2(p_1 p_2)} f_{AP}(z)\, d\sigma_{PT}^{(n-1)} \tag{5.11}$$

with:

$$\chi(z, s_{ij}) = \frac{(1+R)(1 + z^4 + (1-z)^4}{R + z^4 + (1-z)^4} \tag{5.12}$$

By equating Equations (5.11) and (5.5) we therefore obtain:

$$d\sigma^{(n)}_{full} \;=\; d\sigma^{(n)}_{PT}\,\chi(z,s_{ij})\,\frac{d\sigma^{(n-1)}_{full}}{d\sigma^{(n-1)}_{PT}} \tag{5.13}$$

Maxwell suggested that while the W-W approximation is not very good unless we are very close to a collinear pole, Equation (5.13) is rather good throughout phase-space, provided we perform the factorization considering the pair of partons with the minimum s_{ij}. In other words, while the value of the full differential cross section is not well reproduced in the W-W approximation away from the collinear poles, what is well approximated is the relative weight of different helicity amplitudes.

Since $d\sigma^{(5)}_{full} = d\sigma^{(5)}_{PT}$, for $n = 6$ we obtain:

$$d\sigma^{(6)}_{full} \;=\; d\sigma^{(6)}_{PT}\chi(z,s_{ij}) \tag{5.14}$$

while for larger n the infrared reduction can be iterated, giving:

$$d\sigma^{(n)}_{full} \;=\; d\sigma^{(n)}_{PT}\prod_{k=6}^{n}\chi_k(z_k,s_{ij}), \tag{5.15}$$

with an obvious notation.

If the two partons which minimize $|s_{ij}|$ belong to initial and final state, we can still use Equations (5.13) and (5.12) provided we keep all of the momenta as outgoing (which implies that the energies of the initial state particles will be negative) and define:

$$z \;=\; \frac{E_i}{E_i+E_j} \tag{5.16}$$

The z defined in this way cannot be interpreted directly as the fraction of momentum anymore, since it will not satisfy the constraint $0 < z < 1$. In particular, if i is the final state parton then $z < 0$, while if i is the initial state, then $z > 1$. However it can be easily checked that with this prescription Equations (5.13) and (5.12) reproduce the desired factorization properties.

This reduction technique has been applied to many processes, see Mangano and Parke[2] for references.

6 Conclusion

In following lectures I have shown how to use the Helicity Amplitude technique to calculate the cross section for many processes involving large numbers of partons in the final state. This technique gives exact analytic matrix elements, numerical recursive algorithms or approximate matrix elements for evaluating cross sections and hence is an extremely powerful tool for phenomenological studies of multi-jet phenomena in High Energy Colliders.

I would like to thank all the organizers of this school for the very warm hospitality provided during my stay in Mexico and to the students who provided many stimulating conversation between sessions.

A Appendix: Spinor Calculus and Feynman Rules

A.1 Spinor Calculus

In the Weyl basis define

$$|p\pm\rangle \equiv \frac{1}{2}(1 \pm \gamma_5)v(p) \qquad \langle p\pm| \equiv \bar{u}(p)\frac{1}{2}(1 \mp \gamma_5). \tag{A.1}$$

From these spinors we can form various spinor products

$$\langle pq\rangle \equiv \langle p-|q+\rangle \qquad [pq] \equiv \langle p+|q-\rangle, \tag{A.2}$$

$$\langle p-|q-\rangle = 0 \qquad \langle p+|q+\rangle = 0 \tag{A.3}$$

which are normalized so that

$$\langle p\pm|\gamma_\mu|p\pm\rangle = 2p_\mu. \tag{A.4}$$

From the properties of the Dirac algebra, it is straightforward to prove the following useful identities:

$$\hat{p}\,|p\pm\rangle = \langle p\pm|\,\hat{p} = 0 \tag{A.5}$$

$$\langle pq\rangle = -\langle qp\rangle, \qquad [pq] = -[qp] \tag{A.6}$$

$$\langle pp\rangle = [pp] = 0 \tag{A.7}$$

$$\hat{p} = |p+\rangle\langle p+| + |p-\rangle\langle p-| \tag{A.8}$$

$$2\,|p\pm\rangle\langle q\pm| = \frac{1}{2}(1 \pm \gamma_5)\gamma^\mu\langle q\pm|\gamma_\mu|p\pm\rangle, \tag{A.9}$$

$$[pq] = -sign(p\cdot q)\langle pq\rangle^* \tag{A.10}$$

$$\langle pq\rangle[qp] \;=\; S_{pq} \;=\; 2(p\cdot q), \tag{A.11}$$

$$\langle p\pm|\,\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}\,|q\pm\rangle \;=\; \langle q\mp|\,\gamma_{\mu_{2n+1}}\cdots\gamma_{\mu_1}\,|p\mp\rangle, \tag{A.12}$$

$$\langle p\pm|\,\gamma_{\mu_1}\cdots\gamma_{\mu_{2n}}\,|q\mp\rangle \;=\; -\langle q\pm|\,\gamma_{\mu_{2n}}\cdots\gamma_{\mu_1}\,|p\mp\rangle, \tag{A.13}$$

$$\langle AB\rangle\langle CD\rangle = \langle AD\rangle\langle CB\rangle + \langle AC\rangle\langle BD\rangle \tag{A.14}$$

$$\langle A+|\,\gamma_\mu\,|B+\rangle\langle C-|\,\gamma^\mu\,|D-\rangle = 2[AD]\langle CB\rangle. \tag{A.15}$$

A numerical recipe for evaluating these spinor products is

$$\langle ij\rangle \equiv \sqrt{|S_{ij}|}\;\exp(i\phi_{ij}), \tag{A.16}$$

$$[ij] \equiv \sqrt{|S_{ij}|}\;\exp(i\bar{\phi}_{ij}). \tag{A.17}$$

If both momenta having positive energy, the phase factor ϕ_{ij} is defined, in a popular representation of the gamma matrices, by

$$\begin{aligned}\cos\phi_{ij} &= \frac{(p_i^1 p_j^+ \;-\; p_j^1 p_i^+)}{\sqrt{p_i^+ p_j^+ S_{ij}}}\\ \sin\phi_{ij} &= \frac{(p_i^2 p_j^+ \;-\; p_j^2 p_i^+)}{\sqrt{p_i^+ p_j^+ S_{ij}}}.\end{aligned} \tag{A.18}$$

Where $p^\pm = (p^0 \;\pm\; p^3)$ and since $p_i^2 = 0$, the spinor product for this representation of gamma matrices are undefined for a momentum vector in the minus 3 direction. If one or more of the momenta in $\langle ij\rangle$ have negative energy, ϕ_{ij} is calculated with minus the momenta with negative energy and then $n\pi/2$ is added to ϕ_{ij} where n is the number of negative momenta in the spinor product. The associated phase factor, $\bar{\phi}_{ij}$, for $[ij]$ can be calculated from S_{ij} using the identity $S_{ij} \equiv \langle ij\rangle\;[ji]$.

A.2 Summary of Feynman Rules

Here we summarize the Color truncated Feynman rules, where all the vertices are cyclically ordered and all momenta are outgoing. Demonstrating that the subamplitudes $m(g_1, g_2, g_3, g_4)$ and $m(q, g_1, g_2, \bar{q})$ are gauge invariant is an easy way to check the consistency of our conventions.

- External, outgoing fermion, F, helicity $\pm$:

$$\langle F \pm | . \tag{A.19}$$

- External, outgoing anti-fermion, $\overline{F}$, helicity $\pm$:

$$|\overline{F} \mp\rangle . \tag{A.20}$$

- External, outgoing vector, momentum p, reference k, helicity $\pm$:

$$\epsilon_\mu^{\pm}(p,k) = \pm \frac{\langle p \pm | \gamma_\mu | k \pm \rangle}{\sqrt{2}\langle k \mp | p \pm \rangle}, \tag{A.21}$$

$$\epsilon^{\pm}(p,k) \cdot \gamma = \pm \frac{\sqrt{2}}{\langle k \mp | p \pm \rangle} \left(|p \mp\rangle \langle k \mp | + |k \pm\rangle \langle p \pm | \right). \tag{A.22}$$

- Fermion propagator. momentum q, in the direction of the fermion arrow:

$$i \, \frac{\hat{q}}{q^2}. \tag{A.23}$$

- Vector propagator, momentum p:

$$-i \, \frac{g_{\mu\nu}}{p^2}. \tag{A.24}$$

- Fermion-vector-antifermion vertex, order $(FV\overline{F})$:

$$i \, \frac{g}{\sqrt{2}} \, \gamma_\mu . \tag{A.25}$$

- Tri-Vector vertex, order (123), all momenta outgoing from vertex:

$$i \, \frac{g}{\sqrt{2}} \left[(p_1 - p_2)_{\mu_3} g_{\mu_1\mu_2} + (p_2 - p_3)_{\mu_1} g_{\mu_2\mu_3} + (p_3 - p_1)_{\mu_2} g_{\mu_3\mu_1} \right]. \tag{A.26}$$

- Quartic-Vector vertex, order (1234):

$$i \frac{g^2}{2} \left(2 g_{\mu_1\mu_3} g_{\mu_2\mu_4} - g_{\mu_1\mu_4} g_{\mu_2\mu_3} - g_{\mu_1\mu_2} g_{\mu_3\mu_4} \right). \tag{A.27}$$

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